

Dipole and Quadrupole Skyrmions in $S=1$ (Pseudo)Spin Systems

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In terms of spin coherent states we have investigated topological defects in 2D $S = 1$ (pseudo)spin quantum system with the bilinear and biquadratic isotropic exchange in the continuum limit. The proper Hamiltonian of the model can be written as bilinear in the generators of $SU(3)$ group (Gell-Mann matrices). The knowledge of such group structure allows us to obtain some new exact analytical results. Analysing the proper classical model we arrive at different skyrmionic solutions with finite energy and the spatial distribution of spin-dipole and/or spin-quadrupole moments termed as dipole, quadrupole, and dipole-quadrupole skyrmions, respectively. Among the latter we would like note the in-plane vortices with the in-plane distribution of spin moment, varying spin length, and the non-trivial distribution of spin-quadrupole moments.

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I. INTRODUCTION

Different topological defects play an important role both in low-energy (spin excitations, domain walls, superfluidity/superconductivity) and high-energy physics from heavy ion collisions to cosmological scenarios^{1–3}. Theoretical approach to its description traditional for strongly correlated systems like quantum (pseudo)spin ones, starts from either (pseudo)spin Hamiltonian with subsequent reduction to either classical models with solutions like in-plane or out-of-plane vortices, and skyrmions. The latter represent the solutions of non-linear σ -model with classical 2D Hamiltonian

$$H_0 = J \int d^2\vec{r} \left[\sum_{i=1}^3 (\vec{\nabla} n_i)^2 \right] \quad (1)$$

for the vector field $\vec{n}(\vec{r}) = \{\sin \theta \cos \Phi, \sin \theta \sin \Phi, \cos \theta\}$, obtained by Belavin and Polyakov⁴ more than two decades ago. A renewed interest to these unconventional spin textures is stimulated by high- T_c problem in doped quasi-2D-cuprates and quantum Hall effect.

The skyrmion spin texture consists of a vortex-like arrangement of the in-plane components of spin with the z -component reversed in the centre of the skyrmion and gradually increasing to match the homogeneous background at infinity. The spin distribution within classical skyrmion is given as follows

$$\Phi = q\varphi \quad \cos \theta = \frac{r^{2q} - \lambda^{2q}}{r^{2q} + \lambda^{2q}}, \quad (2)$$

or for $q = 1$

$$n_x = \frac{2r\lambda}{r^2 + \lambda^2} \cos \varphi, \quad n_y = \frac{2r\lambda}{r^2 + \lambda^2} \sin \varphi, \quad n_z = \frac{r^2 - \lambda^2}{r^2 + \lambda^2}. \quad (3)$$

In terms of the stereographic variables the skyrmion with radius λ and phase φ_0 centered at a point z_0 is identified with spin distribution $w(z) = \frac{\Lambda}{z - z_0}$, where $z = x + iy = re^{i\varphi}$ is a point in the complex plane, $\Lambda = \lambda e^{i\theta}$, and characterized by three modes: translational, or positional z_0 -mode, "rotational" θ -mode and "dilatational" λ -mode. Each of them corresponds to a certain symmetry of the classical skyrmion configuration. For example, θ -mode

corresponds to a combination of rotational symmetry and internal U(1) transformation. Classical skyrmionic energy $E_q = 8\pi qJS^2$ is proportional to its topological charge and does not depend on its radius.

Other well known solutions of isotropic and anisotropic 2D Heisenberg model are the in-plane and out-of-plane vortices⁵⁻⁷ which have the energy logarithmically dependent on the size of the system. The in-plane vortex is described by the formulas $\Phi = q\varphi$, $\cos\theta = 0$. The $\theta(r)$ dependence for the out-of-plane vortex cannot be found analytically.

Non-linear σ -model can be addressed as classical continuum limit of 2D Heisenberg ferromagnet with isotropic spin-Hamiltonian

$$H = -J \sum_{i,b} \hat{\vec{S}}_i \hat{\vec{S}}_{i+b}. \quad (4)$$

The simplest quantum generalization of skyrmionic solutions could be obtained in frames of spin coherent states⁸, where the wave function of the quantum spin system, which maximally corresponds to classical skyrmion, is a product of spin coherent states. In the case of spin $s = \frac{1}{2}$

$$\Psi_{sk}(0) = \prod_i [\cos \frac{\theta_i}{2} e^{i\frac{\varphi_i}{2}} |\uparrow\rangle + \sin \frac{\theta_i}{2} e^{-i\frac{\varphi_i}{2}} |\downarrow\rangle], \quad (5)$$

where $\theta_i = \arccos[(r_i^2 - \lambda^2)/(r_i^2 + \lambda^2)]$. The coherent state implies a maximal equivalence to the classical one with the minimal uncertainty of spin components. Actually, every on-site spin in a lattice is assumed to be subjected to a molecular field $\vec{H}(\vec{r}) \propto \vec{n}(\vec{r}) = \{\sin\theta \cos\Phi, \sin\theta \sin\Phi, \cos\theta\}$ which spatial distribution forms a skyrmionic texture.

The coherent state approach appears to be rather simple for the $s = 1/2$ spin systems. Indeed, on the one hand, spin-Hamiltonian for $s = 1/2$ quantum system is restricted to have isotropic, or anisotropic bilinear Heisenberg exchange form like (4). On the other hand, the trial wave function (5) is simply parameterized by the vector field $\vec{n}(\vec{r})$. Some quasiparticle properties of quantized skyrmion in the $s = 1/2$ model are addressed in⁹.

The situation becomes more involved for the $S \geq 1$ (pseudo)spin systems, where, generally speaking, we have to deal with additional non-Heisenberg terms in (pseudo)spin-Hamiltonian and several vector fields to parameterize the trial wave function like (5). A

principal difference between the $S = \frac{1}{2}$ and $S \geq 1$ quantum systems lies in what concerns the order parameters. The only single-site order parameter in the former case is an average spin (dipole) moment $\langle S_{x,y,z} \rangle$, whereas in the latter one has additional "spin-multipole" parameters like "spin-quadrupole" (spin nematic) averages $\langle \{\hat{S}_i \hat{S}_j\} \rangle$, where $\{\hat{S}_i \hat{S}_j\} = \hat{S}_i \hat{S}_j + \hat{S}_j \hat{S}_i$. Hence, we may expect in $S = 1$ quantum spin systems different topological defects with spatial distribution of not only nonzero spin (dipole) moment (dipole skyrmions), or spin-quadrupole moment (quadrupole skyrmions), but more involved dipole-quadrupole skyrmions with a nonzero distribution of both order parameters.

Interestingly, that in a sense, the $s = 1/2$ quantum spin system is closer to the classical one ($S \rightarrow \infty$) also characterized by the single-site vector order parameter than, for instance, the $S = 1$ quantum spin system with its eight site order parameters. In a whole, we should expect for the $S \geq 1$ (pseudo)spin systems an appearance both of unconventional topology and the complicated order parameter textures¹⁰.

The main interest in $S = 1$ quantum spin systems is provoked by the $S = 1$ quantum spin chains displaying the Haldane gap¹¹. Several special examples of the 2D $S = 1$ spin systems have been extensively discussed earlier in connection with the study of the static and dynamic properties of anisotropic Heisenberg 2D magnets with a single-ion anisotropy (see *e.g.*^{12,13}). Some aspects of the topological structure of vortices in 2D $S = 1$ systems were discussed by different authors in connection with ^3He problem^{2,3,10} and triplet superconductivity¹⁴. The quantum "spin nematic" phase of the fully isotropic $S = 1$ system was addressed recently by Ivanov and Kolezhuk¹⁵.

Our interest in this field was motivated by a problem of a description of soliton-like excitations in quasi-2D cuprates in frames of a novel scenario proposed by one of authors¹⁶. The model considers doped cuprates as a system of singlet local bosons moving in a lattice formed by hole centers CuO_4^{5-} . Such a center has a complex ground state multiplet $^1A_{1g}-^1E_u$ which can be described by a pseudo-spin $S = 1$.

In this paper we make use of the spin coherent (SC-) state approach to describe the skyrmion-like topological defects in $S = 1$ quantum (pseudo)spin 2D systems with isotropic

non-Heisenberg spin-Hamiltonian. These solutions are a special case of so-called CP_N spinors discussed in Refs.^{17,18}, though its authors focused their interest only in CP_1 and CP_3 models, rather than the CP_2 spinors of our model. Our interest is mainly focused on the unconventional quadrupole and dipole-quadrupole skyrmion-like static solutions.

In Sec.II we address the isotropic bilinear-biquadratic Hamiltonian for the $S = 1$ quantum (pseudo)spin systems, the parameterization of the trial wave function, the $\text{SU}(3)$ -model approach, and the reduction procedure to the classical continuum limit of the $S = 1$ model. In Sec.III the unconventional skyrmion-like solutions are analyzed, including the known magnetic (dipole) skyrmion¹⁴, and unusual quadrupole and dipole-quadrupole skyrmions with a non-trivial spatial distribution of dipole and/or quadrupole (pseudo)spin order parameters.

II. CLASSICAL DESCRIPTION OF THE $S = 1$ QUANTUM (PSEUDO)SPIN SYSTEMS

In general, isotropic non-Heisenberg spin-Hamiltonian for the $S = 1$ quantum (pseudo)spin systems should include both bilinear Heisenberg exchange term and biquadratic non-Heisenberg exchange term:

$$\begin{aligned}\hat{H} &= -\tilde{J}_1 \sum_{i,\eta} \hat{\vec{S}}_i \hat{\vec{S}}_{i+\eta} - \tilde{J}_2 \sum_{i,\eta} (\hat{\vec{S}}_i \hat{\vec{S}}_{i+\eta})^2 = \\ &= -J_1 \sum_{i,\eta} \hat{\vec{S}}_i \hat{\vec{S}}_{i+\eta} - J_2 \sum_{i,\eta} \sum_{k \geq j}^3 (\{\hat{S}_k \hat{S}_j\}_i \{\hat{S}_k \hat{S}_j\}_{i+\eta})\end{aligned}\tag{6}$$

where J_i are the appropriate exchange integrals, $J_1 = \tilde{J}_1 - \tilde{J}_2/2$, $J_2 = \tilde{J}_2/2$, i and η denote the summation over lattice sites and nearest neighbours, respectively. In our spin-1 model we use trial functions

$$\psi = \prod_{j \in \text{lattice}} c_i(j) \psi_i = \prod_{j \in \text{lattice}} (a_i(j) + ib_i(j)) \psi_i\tag{7}$$

Here j labels a lattice site and the spin functions ψ_i in cartesian basis are used: $\psi_z = |10\rangle$ and $\psi_{x,y} \sim (|11\rangle \pm |1-1\rangle)/\sqrt{2}$. The linear (dipole) spin-operator is represented by simple

matrix:

$$\langle \psi_i | S_j | \psi_k \rangle = -i\varepsilon_{ijk},$$

and for the order parameters one easily obtains:

$$\langle \hat{\vec{S}} \rangle = -2[\vec{a}, \vec{b}], \quad \langle \{\hat{S}_i \hat{S}_j\} \rangle = 2(\delta_{ij} - a_i a_j - b_i b_j) \quad (8)$$

given the normalization constraint $\vec{a}^2 + \vec{b}^2 = 1$. Thus, for the case of spin-1 system the order parameters are determined by two classical vectors (two real components of one complex vector $\vec{c} = \vec{a} + i\vec{b}$ from (7)). The two vectors are coupled, so the minimal number of dynamic variables describing the $S = 1$ spin system appears to be equal to four (see Ref.¹² and Sec.2.2 below). Hereafter we would like to emphasize the *director* nature of the \vec{c} vector field: $\psi(\vec{c})$ and $\psi(-\vec{c})$ describe the physically identical states.

The dipole or magnetic skyrmions in the spin-1 systems were addressed in the paper¹⁴. The structure of the order parameter admits the existence of more general types of solutions which are purely quadrupole ("electric") or mixed dipole-quadrupole ("magneto-electric") ones. One should note that, in common, the length of the spin vector in S=1 model must not be fixed. The order parameters structure is responsible for another important property of the S=1 systems: it allows the existence of more than one topological quantum number.

A. SU(3)-symmetry model: Gell-Mann operators and effective Hamiltonian for S=1 model

Three spin-linear (dipole) operators $\hat{S}_{1,2,3}$ and five independent spin-quadrupole operators $\{\hat{S}_i, \hat{S}_j\} - \frac{2}{3}\hat{\vec{S}}^2 \delta_{ij}$ at $S = 1$ form eight Gell-Mann operators being the generators of the SU(3) group. Below we will make use of the appropriate Gell-Mann 3×3 matrices $\Lambda^{(k)}$, which differ from the conventional $\lambda^{(k)}$ only by a renumeration: $\lambda^{(1)} = \Lambda^{(6)}$, $\lambda^{(2)} = \Lambda^{(3)}$, $\lambda^{(3)} = \Lambda^{(8)}$, $\lambda^{(4)} = \Lambda^{(5)}$, $\lambda^{(5)} = -\Lambda^{(2)}$, $\lambda^{(6)} = \Lambda^{(4)}$, $\lambda^{(7)} = \Lambda^{(1)}$, $\lambda^{(8)} = \Lambda^{(7)}$. First three matrices $\Lambda^{(1,2,3)}$

correspond to linear (dipole) spin operators:

$$\Lambda^{(1)} = S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \Lambda^{(2)} = S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}; \quad \Lambda^{(3)} = S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

while other five matrices correspond to quadratic (quadrupole) spin operators:

$$\begin{aligned} \Lambda^{(4)} = -\{S_z S_y\} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \Lambda^{(5)} = -\{S_x S_z\} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \Lambda^{(6)} = -\{S_x S_y\} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ \Lambda^{(7)} = -\frac{1}{\sqrt{3}}(S_x^2 + S_y^2 - 2S_z^2) &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}; \quad \Lambda^{(8)} = S_y^2 - S_x^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ S_x^2 + S_y^2 + S_z^2 &= 2\hat{E} \end{aligned}$$

with \hat{E} being a unit 3×3 matrix.

The generalized spin-1 model can be described by the Hamiltonian bilinear on the SU(3)-generators $\Lambda^{(k)}$ ¹⁹

$$\hat{H} = - \sum_{i, i+\eta} \sum_{k=1}^8 J_{km} \hat{\Lambda}_i^{(k)} \hat{\Lambda}_{i+\eta}^{(m)}.$$

Here i, η denote lattice sites and nearest neighbors, respectively. This is a $S = 1$ counterpart of the $S = 1/2$ model Heisenberg Hamiltonian with three generators of the SU(2) group or Pauli matrices included instead of eight Gell-Mann matrices.

In frames of isotropic bilinear-biquadratic model we study in this work the 8×8 matrix J_{km} is assumed to be diagonal with elements $J_{11} = J_{22} = J_{33} = J_1$, and $J_{44} = J_{55} = \dots = J_{88} = J_2$. Fully isotropic SU(3) model with $J_1 = J_2$ corresponds to a ferromagnetic version of so-called Uimin-Lai-Sutherland model¹⁹. It should be noted that the isotropic in a real space Hamiltonian with $J_1 \neq J_2$ can be considered as the anisotropic one in the 8-dimensional SU(3) group space, and the symmetry of the model breaks to the subgroup $\text{SO}(3) \subset \text{SU}(3)$. In other words, the breaking of the condition $J_1 = J_2 (= J)$ can be considered as an appearance of the exchange anisotropy in the 8-dimensional phase space. To describe

this anisotropy one might introduce the ratio $\lambda = J_2/J_1$. Hereafter, we shall associate the limiting cases $\lambda = 0$ and $\lambda \rightarrow \infty$ with purely magnetic and electric solutions, respectively. Such an effective anisotropy in $S = 1$ systems differs strongly from the real spatial exchange anisotropy in $S = 1/2$ systems which results in preferred spin orientations.

If one considers the magnetic field parallel to z-axis or the anisotropy of exchange parameters in a real space the symmetry breaks to $SO(2) \subset SO(3)$. However, given the definite relations between the anisotropy constants and exchange integrals this model can be reduced to spin-1/2 isotropic model. This case merits the separate examination.

B. The classical continuum limit of S=1 model

Having substituted our trial wave function (7) to $\langle \hat{H} \rangle$ provided $\langle \hat{\vec{S}}(1) \hat{\vec{S}}(2) \rangle = \langle \hat{\vec{S}}(1) \rangle \langle \hat{\vec{S}}(2) \rangle$ we arrive at the Hamiltonian of the isotropic classical spin-1 model in the continuum approximation as follows:

$$H = J_1 \int d^2\vec{r} \left[\sum_{i=1}^3 (\vec{\nabla} S_i)^2 \right] + J_2 \int d^2\vec{r} \left[\sum_{i,j=1}^3 (\vec{\nabla} a_i a_j + \vec{\nabla} b_i b_j)^2 \right] + \frac{4(J_2 - J_1)}{c^2} \int |\vec{S}|^2 d^2\vec{r}, \quad (9)$$

where $\vec{S} = -2[\vec{a} * \vec{b}] = \langle \hat{\vec{S}} \rangle$. The effective exchange anisotropy defines the third "gradientless" term in the Hamiltonian that breaks the scaling invariance of the model. Such an effect in $S = 1/2$ system appears due to the real spatial exchange anisotropy which defines the magnetic length⁷.

The spin-1 model differs from the spin-1/2 model due to the appearance of the additional ("nonmagnetic") degrees of freedom. When $\langle \hat{\vec{S}} \rangle = 0$ (e.g., if one of \vec{a}, \vec{b} vectors turns into zero) we have a non-zero part of classical Hamiltonian (proportional to J_2) and can get the nontrivial configurations of non-zero vector. We shall call this configuration "electric skyrmion". It should be described by one vector with the fixed length, so the topological classification of such solutions is completely analogous to that of the spin-1/2 classical solutions, which are described by the order parameter being a fixed-length spin vector. Below, we shall study this solution, deriving it via reducing our biquadratic model to the non-linear

O(3)-model. The topological charge of the classical electric skyrmion with $\vec{b} = 0$ can be defined by usual formula as follows

$$Q = \frac{1}{8\pi} \int d^2\vec{r} \varepsilon_{\nu\mu} (\vec{a} * [\vec{a}_\nu * \vec{a}_\mu]) = \frac{1}{4\pi} \int r dr d\varphi (\vec{a} * [\vec{a}_r * \frac{1}{r} \vec{a}_\varphi]), \quad (10)$$

where the subscripts denote derivative.

In the continuum limit for $J_1 = J_2 = J$ the Hamiltonian (9) can be transformed into the classical Hamiltonian of the $SU(3)$ -symmetric scale-invariant model:

$$H = \frac{1}{2} J \int d^2\vec{r} [\sum_{k=1}^8 (\vec{\nabla} \bar{\psi} \hat{\Lambda}^{(k)} \psi)^2].$$

where we make use of the single-site wave function in the form as follows:

$$\psi = \begin{pmatrix} R_1 \exp(i\Phi_1) \\ R_2 \exp(i\Phi_2) \\ R_3 \exp(i\Phi_3) \end{pmatrix}; \quad |\vec{R}|^2 = 1, \quad (11)$$

with $\vec{R} = \{\sin \Theta \cos \eta, \sin \Theta \sin \eta, \cos \Theta\}$. In accordance with the *director* nature of the \vec{a}, \vec{b} vector fields we have to vary the angles Θ, η, Φ_i in the range $(0, \pi)$. The Hamiltonian can be rewritten as follows:

$$H_{isotr} = 2J \int d^2\vec{r} \{ (\vec{\nabla} \Theta)^2 + \sin^2 \Theta (\vec{\nabla} \eta)^2 + \\ + \sin^2 \Theta \cos^2 \Theta [\cos^2 \eta (\vec{\nabla} \Psi_1)^2 + \sin^2 \eta (\vec{\nabla} \Psi_2)^2] + \sin^4 \Theta \cos^2 \eta \sin^2 \eta (\vec{\nabla} \Psi_1 - \vec{\nabla} \Psi_2)^2 \}, \quad (12)$$

where we have introduced $\Psi_1 = \Phi_1 - \Phi_3, \Psi_2 = \Phi_3 - \Phi_2$. For an isolated (pseudo)spin system the Φ_3 phase is arbitrary, hence the minimal number of dynamic variables describing the $S = 1$ (pseudo)spin system equals to four ($=4S^{12}$). However, for a more general situation, when the (pseudo)spin system represents only the part of the bigger system, and we are forced to consider the coupling with the additional degrees of freedom, the Φ_3 phase turns into a non-trivial parameter: $\Phi_3 = Q_3 \varphi$. The topological solutions for our Hamiltonian (12) can be classified at least by three topological quantum numbers (winding numbers): phases $\eta, \Psi_{1,2}$ can change by 2π after the passing around the center of the defect. It should be noted

that the *director* nature of the \vec{a}, \vec{b} vector fields implies the possibility for winding numbers to take half-integer values. The appropriate modes may have very complicated topological structure due to the possibility for one defect to have several different centers (while one of the phases $\eta, \Psi_{1,2,3}$ changes by 2π given one turnover around one center (r_1, φ_1) , other phases may pass around other centers (r_i, φ_i)). Each center in a multi-center defect can be considered as a quasiparticle. In this connection it should be noted that the spin-1 model differs from the spin-1/2 one by the fact that the former in common assumes quasiparticles of different types due to the existence of different topological quantum numbers for different centers.

Finally, it should be noted that the above model approach can be extended to the $S > 1$ isotropic (pseudo)spin systems. In the case of spin S one has to calculate averages like $\langle S_{i_1} \dots S_{i_n} \rangle$ where $n = 1, \dots, 2S$. One can suggest that in our Hamiltonian the combinations $c_{i_1} \dots c_{i_n}$ with a fixed-length vector \vec{c} should appear. Now let us notice that for any $k \in N$ the model with discrete Hamiltonian

$$H_{kk} = -J_{kk} \sum_{i, \eta} \sum_{j_1 \dots j_k=1}^3 \left[\prod_{q=1}^k c_{j_q}(i) c_{j_q}(i + \eta) \right] \quad (13)$$

in the continuum limit can be reduced to

$$H_k = J_k \int d^2 \vec{r} \left[\sum_{j_1 \dots j_k=1}^3 (\vec{\nabla} c_{j_1} \dots c_{j_k})^2 \right]. \quad (14)$$

Making use of $\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$ provided $|\vec{c}| = \text{const}$, we come to

$$H_k = k J_k |\vec{c}|^{2k-2} \int d^2 \vec{r} \left[\sum_{i=1}^3 (\vec{\nabla} c_i)^2 \right]. \quad (15)$$

It is a non-linear $O(3)$ -model. Its skyrmionic solutions differ from the conventional ones by the energy, due to the term $k|\vec{c}|^{2k-2}$. In our spin-1 case $k = 2$, $\vec{c} = \vec{n}$. So, if the reduction of spin- S quantum model to the quasi-classical one (13) will be made by means of some parameterization of the trial wavefunction it will be easy to obtain the skyrmionic solutions of the model with finite energy $k J_k q |\vec{c}|^{2k}$.

III. UNCONVENTIONAL SKYRMIONS IN $S = 1$ (PSEUDO)SPIN SYSTEMS

A. Dipole skyrmions

One important case of the spin-1 model when $J_2 = 0$ (purely Heisenberg Hamiltonian) also has skyrmionic solutions, which were found earlier in Ref.¹⁴. When the \vec{a}, \vec{b} vectors are perpendicular to each other ($\vec{a} \perp \vec{b}$), the model also reduces to the nonlinear O(3)-model. The solution from Ref.¹⁴ is described by the following formulas (in polar coordinates):

$$\begin{aligned}\sqrt{2}\vec{a} &= (\vec{e}_z \sin \theta - \vec{e}_r \cos \theta) \sin \varphi + \vec{e}_\varphi \cos \varphi ; \\ \sqrt{2}\vec{b} &= (\vec{e}_z \sin \theta - \vec{e}_r \cos \theta) \cos \varphi - \vec{e}_\varphi \sin \varphi .\end{aligned}\tag{16}$$

The fixed-length spin vector is distributed in the same way as in the conventional skyrmion with topological charge $q = 1$ (2). However, unlike the usual skyrmions, the solutions (16) have additional topological structure due to the existence of two vectors \vec{a} and \vec{b} . Going around the center of the defect the vectors \vec{a} and \vec{b} can make N turns around the spin vector $\propto [\vec{a} \times \vec{b}]$. Thus, we can introduce two topological quantum numbers: N and q ¹⁴. In addition, it should be noted that q number may be half-integer. Hereafter we shall call the skyrmionic solutions with the only non-zero magnetic component as the dipole or magnetic ones.

B. Quadrupole skyrmions

Magnetic skyrmions as the solutions of purely Heisenberg (pseudo)spin Hamiltonian¹⁴ were obtained given the restriction $\vec{a} \perp \vec{b}$ and the lengths of these vectors were fixed. These restrictions lead to the pure magnetic solution and enabled to use a subgroup for the topological classification¹⁴. It was SO(3) which is generated by $\hat{\Lambda}^{(1)}, \hat{\Lambda}^{(2)}, \hat{\Lambda}^{(3)}$ matrices forming the Heisenberg bilinear term.

Hereafter we address another situation with purely biquadratic (pseudo)spin Hamiltonian ($J_1=0$) and treat the non-magnetic (electric) degrees of freedom. The topological classifi-

cation of the purely electric solutions is simple because it is also based on the making use of subgroup instead of the full group. We address the solutions given $\vec{a} \parallel \vec{b}$ and the fixed lengths of the vectors, so we use for the classification the same subgroup as above.

The biquadratic part of the Hamiltonian

$$H_{biq} = J_2 \int d^2\vec{r} \left[\sum_{i,j=1}^3 (\vec{\nabla} a_i a_j + \vec{\nabla} b_i b_j)^2 \right] \quad (17)$$

can be rewritten as follows

$$H_{biq} = J_2 \int d^2\vec{r} \left[\sum_{i,j=1}^3 (\vec{\nabla} n_i n_j)^2 \right], \quad (18)$$

where $\vec{a} = \alpha \vec{n}$, $\vec{b} = \beta \vec{n}$, and $\alpha + i\beta = \exp(i\kappa)$, $\kappa \in R$. We denote $\vec{n} = n\{\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta\}$. Using simple formula $\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$ together with the normalization constraint $|\vec{n}|^2 = \text{const}$, we reduce the expression for H_{biq} to the familiar nonlinear O(3)-model:

$$H_{biq} = 2J_2 |\vec{n}|^2 \int d^2\vec{r} \left[\sum_{i=1}^3 (\vec{\nabla} n_i)^2 \right]. \quad (19)$$

Its solutions are skyrmions, but instead of the spin distribution in magnetic skyrmion we have solutions with zero spin, but the non-zero distribution of five spin-quadrupole moments $\langle \Lambda^{(4,5,6,7,8)} \rangle$, or $\langle \{S_i S_j\} \rangle$ which in turn are determined by the distribution of the $\vec{a}(\vec{n})$ vector:

$$\Phi = q\varphi + \Phi_0; \quad \cos \Theta = \frac{r^{2q} - \lambda^{2q}}{r^{2q} + \lambda^{2q}} \quad (20)$$

with a classical skyrmion energy

$$E_{el} = 16\pi q J_2. \quad (21)$$

The distribution of the spin-quadrupole moments $\langle \{S_i S_j\} \rangle$ can be easily obtained:

$$\begin{aligned} \langle S_x^2 \rangle &= \frac{(r^{2q} + \lambda^{2q})^2 \cos^2 q\varphi + (r^{2q} - \lambda^{2q})^2 \sin^2 q\varphi}{(r^{2q} + \lambda^{2q})^2}; \\ \langle S_y^2 \rangle &= \frac{(r^{2q} + \lambda^{2q})^2 \sin^2 q\varphi + (r^{2q} - \lambda^{2q})^2 \cos^2 q\varphi}{(r^{2q} + \lambda^{2q})^2}; \end{aligned}$$

$$\begin{aligned}
\langle \{S_x S_y\} \rangle &= -\frac{2r^{2q}\lambda^{2q}\sin 2q\varphi}{(r^{2q} + \lambda^{2q})^2}; & \langle S_z^2 \rangle &= \frac{4r^{2q}\lambda^{2q}}{(r^{2q} + \lambda^{2q})^2}; \\
\langle \{S_x S_z\} \rangle &= \frac{2(\lambda^{2q} - r^{2q})r^q\lambda^q\sin q\varphi}{(r^{2q} + \lambda^{2q})^2}; & \langle \{S_y S_z\} \rangle &= \frac{2(\lambda^{2q} - r^{2q})r^q\lambda^q\cos q\varphi}{(r^{2q} + \lambda^{2q})^2}.
\end{aligned} \tag{22}$$

One should be emphasized that the distribution of five independent quadrupole order parameters for the electric skyrmion are straightforwardly determined by a single vector field $\vec{n}(\vec{r})$. The phase factor $\alpha + i\beta$ can be arbitrary, because it is not included in the Hamiltonian. It may be written as follows: $\alpha + i\beta = \exp(ip\varphi)$. Such classes of solutions are physically equivalent in frames of our model.

C. Dipole-quadrupole skyrmions

In this subsection we would like to write out some solutions of fully SU(3)-symmetric isotropic model for (pseudo)spin Hamiltonian (9) with $J_1 = J_2$. We do not derive all the possible solutions of this model because it merits to be a subject of the separate examination. Our main goal now is only to illustrate the richness of the model using as examples the simplest solutions. The solutions of this model may be classified taking into account whether they have each of the winding numbers to be zero or not. Below we will briefly consider two simplest classes of such solutions (the choice of classes is defined by the simplicity of integration).

First of all we shall consider simplest solutions of the equations minimizing energy functional. One type of skyrmions can be obtained given the trivial phases $\Psi_{1,2}$. If these are constant, the \vec{R} vector distribution (see (11)) represents the skyrmion described by the usual formula (2). All but one topological quantum numbers are zero for this class of solutions. It includes both dipole and quadrupole solutions: depending on selected constant phases one can obtain both "electric" and different "magnetic" skyrmions. The substitution $\Phi_1 = \Phi_2 = \Phi_3$ leads to the electric skyrmion which was obtained as a solution of more general SU(3)-anisotropic model in the previous section. Another example can be $\Phi_1 = \Phi_2 = 0, \Phi_3 = \pi/2$.

This substitution implies $\vec{b} \parallel Oz$, $\vec{a} \parallel Oxy$, $\vec{S} \parallel Oxy$, and $\vec{S} = \sin \Theta \cos \Theta \{\sin \eta, -\cos \eta, 0\}$. Nominally, this is the in-plane spin vortex with a varying length of the spin vector

$$|\vec{S}| = \frac{2r\lambda|r^2 - \lambda^2|}{(r^2 + \lambda^2)^2},$$

which turns into zero at the circle $r = \lambda$, at the center $r = 0$ and at the infinity $r \rightarrow \infty$, and has maxima at $r = \lambda(\sqrt{2} \pm 1)$. In addition to the non-zero in-plane components of spin-dipole moment $\langle S_{x,y} \rangle$ (or $\langle \Lambda^{(1,2)} \rangle$) this vortex is characterized by a non-zero distribution of (pseudo)spin-quadrupole moments $\langle \Lambda^{(6,7,8)} \rangle$.

Here we would like to emphasize the difference between spin-1/2 systems in which there are such the solutions as in-plane vortices with the energy having a well-known logarithmic dependence on the size of the system and fixed spin length, and spin-1 systems in which the in-plane vortices also can exist but they may have a finite energy and a varying spin length. The distribution of quadrupole components associated with in-plane spin-1 vortex is non-trivial. Such solutions can be named as "in-plane dipole-quadrupole skyrmions".

Other type of solutions we get given the phases $\Psi_1 = Q_1\varphi$, $\Psi_2 = Q_2\varphi$ with two integer winding numbers $Q_{1,2}$ and $\eta = \eta(r)$, $\Theta = \Theta(r)$. Easiest way to obtain the solutions of variational equations is to put one of these functions to be constant. This way we arrive at six types of solutions. The first one $\Theta = 0$ corresponds to the trivial spinor

$$(i) : \psi = \begin{pmatrix} R_1 \exp(i\Phi_1) \\ R_2 \exp(i\Phi_2) \\ R_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (23)$$

Other solutions being written in complex form with $z = re^{i\varphi}/\lambda$ ($\bar{z} = z^*$) are as follows:

$$(ii) : \Theta = \pi/2 + \pi k, k \in Z, \eta(z) = \arctan[|z|^{(q_1+q_2)}]; \psi = \frac{\exp(iq_1\varphi)}{(1 + |z|^{2(q_1+q_2)})^{1/2}} \begin{pmatrix} 1 \\ (-1)^k \bar{z}^{(q_1+q_2)} \\ 0 \end{pmatrix}; \quad (24)$$

$$(iii) : \eta = \pi k, k \in Z, \Theta(z) = \arctan[|z|^{q_1}]; \psi = \frac{1}{(1 + |z|^{2q_1})^{1/2}} \begin{pmatrix} (-1)^k \bar{z}^{q_1} \\ 0 \\ 1 \end{pmatrix}; \quad (25)$$

$$(iv) : \eta = \pi/2 + \pi k, k \in Z, \Theta(z) = \arctan[|z|^{q_2}]; \psi = \frac{1}{(1 + |z|^{2q_1})^{1/2}} \begin{pmatrix} 0 \\ (-1)^k \bar{z}^{q_2} \\ 1 \end{pmatrix}; \quad (26)$$

$$(v) : \eta = \pi/4 + \pi k/2, k \in Z, q_1 = -q_2 = q, \Theta(z) = \arctan[|z|^q]; \psi = \frac{1}{(1 + |z|^{2q})^{1/2}} \begin{pmatrix} z^q/\sqrt{2} \\ z^q/\sqrt{2} \\ 1 \end{pmatrix}; \quad (27)$$

$$(vi) : \eta = \pi/4 + \pi k/2, k \in Z, q_1 = q_2 = q, \Theta(z) = \arccos\left[\frac{|z|^{2q} - 1}{|z|^{2q} + 1}\right]; \psi = \frac{1}{(1 + |z|^{2q})} \begin{pmatrix} z^q\sqrt{2} \\ \bar{z}^q\sqrt{2} \\ 1 - |z|^{2q} \end{pmatrix}; \quad (28)$$

The energy for solution (vi) is $E_6 = 16\pi qJ$, while for others ((ii) – (v)) $E_k = 4\alpha_k\pi^2 J$ with $\alpha_2 = q_1 + q_2$, $\alpha_3 = q_1$, $\alpha_4 = q_2$ and $\alpha_5 = q$. Similar solutions, as so-called CP_N spinors

$$\eta(z) = \frac{1}{(a^2 + Nr^{2q})^{1/2}} \begin{pmatrix} a \\ z^q \\ z^q \\ \dots \\ \dots \\ z^q \end{pmatrix}, \quad (29)$$

(q is a winding number, a is any constant) are discussed in Ref.^{17,18}, though its authors focused their interest only in CP_1 and CP_3 models, rather than the CP_2 spinors of our model.

Above we address the dipole-quadrupole solutions with the only angular parameter having the $z(r, \varphi)$ -dependence. Hereafter, we consider the nontrivial solution with two winding numbers and unconventional z -dependence of angular parameters. To this end we make use of a ψ -spinor in another form as follows:

$$\psi = \begin{pmatrix} \sin(\frac{\theta}{2}) \exp(i\frac{\alpha}{2})(\cos(\frac{\beta}{2}) \sin(\frac{\Phi}{2}) + i \sin(\frac{\beta}{2}) \cos(\frac{\Phi}{2})) \\ \sin(\frac{\theta}{2}) \exp(i\frac{\alpha}{2})(\sin(\frac{\beta}{2}) \sin(\frac{\Phi}{2}) - i \cos(\frac{\beta}{2}) \cos(\frac{\Phi}{2})) \\ \cos(\frac{\theta}{2}) \end{pmatrix}. \quad (30)$$

In the system where \hat{S}_z is diagonal it takes form

$$\begin{pmatrix} \sin(\frac{\theta}{2}) \cos(\frac{\Phi}{2}) \exp(i\frac{\alpha+\beta}{2}) \\ \sin(\frac{\theta}{2}) \sin(\frac{\Phi}{2}) \exp(i\frac{\alpha-\beta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}.$$

Such a parametrization is opportune when we are interested in solutions with mean-values of S_z and Q_{zz} independent of φ . The density of the energy functional with $J_1 = J_2 = 1$ can be written out as follows:

$$W = \frac{1}{2}[(\vec{\nabla}\theta)^2 + \frac{\sin^2\theta}{4}\{(\vec{\nabla}\alpha)^2 + 2\sin\Phi(\vec{\nabla}\alpha, \vec{\nabla}\beta) + (\vec{\nabla}\beta)^2\} + \sin^2\frac{\theta}{2}\{(\vec{\nabla}\Phi)^2 + \cos^2\Phi\sin^2\frac{\theta}{2}(\vec{\nabla}\beta)^2\}]. \quad (31)$$

The substitution into the equations minimizing this functional of $\alpha = q\varphi$, $\beta = p\varphi$ and $\theta = \theta(r)$, $\Phi = \Phi(r)$ immediately satisfies two of them; the remaining two are as follows:

$$4(\theta_{rr} + \frac{\theta_r}{r}) = \sin\theta\Phi_r^2 + \frac{\sin\theta}{r^2}[p^2\cos^2\Phi + \cos\theta(q + p\sin\Phi)^2];$$

$$\Phi_{rr} + \frac{\Phi_r}{r} = -\text{ctg}\frac{\theta}{2}\theta_r\Phi_r + \frac{p\cos\Phi}{r^2}(q\cos^2\frac{\theta}{2} - p\sin^2\frac{\theta}{2}\sin\Phi). \quad (32)$$

The numbers $\frac{p+q}{2}$ and $\frac{p-q}{2}$ are to be clearly integer. The energy takes the following form:

$$W = \theta_r^2 + \sin^2\frac{\theta}{2}\Phi_r^2 + \frac{\sin^2\theta}{4r^2}[q^2 + 2pq\sin\Phi + p^2] + \frac{p^2}{r^2}\sin^4\frac{\theta}{2}\cos^2\Phi. \quad (33)$$

To find solutions with the finite energy one must choose the conditions $\theta \rightarrow k\pi$ at the infinity with k being integer. The second equation implies that at the infinity $\Phi \rightarrow (2m+1)\pi/2$ when $\theta \rightarrow k\pi$. For the simplest case with minimal different values of the winding numbers: $p=1, q=3$ we have found the exact solution

$$\sin\Phi = \pm \frac{\lambda^2 - r^2}{\lambda^2 + r^2}; \quad \cos\theta(r) = \pm \frac{r^4(l^2 + \lambda^2) - (r^2 + \lambda^2)l^4}{r^4(l^2 + \lambda^2) + (r^2 + \lambda^2)l^4}. \quad (34)$$

The energy of the solution does not depend on two radii λ and l and equals to 16π (half the energy of the electrical skyrmion). The asymptotic behaviour of the spinor is (two cases correspond to two signs for $\cos \theta$) at zero and at the infinity

$$\text{from } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \frac{\exp(2i\varphi)}{\sqrt{2}} \\ \frac{\exp(i\varphi)}{\sqrt{2}} \\ 0 \end{pmatrix}, \text{ or from } \begin{pmatrix} \frac{\exp(2i\varphi)}{\sqrt{2}} \\ \frac{\exp(i\varphi)}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively.

Now let us discuss the solutions from other point of view. In SU(3)-isotropic case the density of energy is invariant under the transformations

$$\hat{T} = \exp[i \sum_{k=1}^8 \gamma_k \hat{\Lambda}^{(k)}],$$

generated by $\{\Lambda^{(k)}\}$, where eight parameters γ_k do not depend on the coordinates. If $\psi_2 = \hat{T}\psi_1$, then

$$\sum_{k=1}^8 (\psi_1 \hat{\Lambda}^{(k)} \psi_1)^2 = \sum_{k=1}^8 (\psi_2 \hat{\Lambda}^{(k)} \psi_2)^2$$

at each point. In other words, both ψ_1 and ψ_2 are the solutions of variational equations with the same energy. Hence, all the solutions can be classified on the irreducible representations of SU(3) group, or supermultiplets. Generally speaking, two spinors ψ_1 and ψ_2 from the same supermultiplet represent only one solution in different parameterizations. For instance, five solutions (ii) – (v) represent one solution $\{\psi_1, \psi_2, 0\}$ which is in fact a well known Belavin-Polyakov solution constructed on two components.

The solutions like electric skyrmion (see Eqs.(20)-(22)) and solution (vi) from this section correspond to another supermultiplet and can be transformed into each other by a rotation around z -axis. Solutions with the \vec{R} vector distribution (23) derived in the beginning of this section can be reduced to electric skyrmion by the transformations generated by $\hat{\Lambda}^{(4)}$ and $\hat{\Lambda}^{(8)}$. Another non-trivial supermultiplet can be constructed from magnetic skyrmion (16). More detailed analysis of the SU(3) supermultiplets will be given elsewhere.

Concluding this subsection we have to notice that the dipole-quadrupole solutions can be obtained also in more general $J_1 \neq J_2$ model but the corresponding equations due to

the term proportional to $J_2 - J_1$ cannot be solved analytically similarly the situation in $S=1/2$ model with exchange anisotropy for the out-of-plane vortices⁵. It should be noted that varying the "anisotropy" parameter $\lambda = J_2/J_1$ from $\lambda = 0$ to $\lambda \rightarrow \infty$ we deal with a transformation of purely magnetic solution to a purely electric one. One might expect that magnetic skyrmion (16) would be stable given $0 \leq \lambda < \lambda_{c1}$, while electric one would be stable given $\lambda_{c2} \leq \lambda < \infty$. In the intermediate range $\lambda_{c1} \leq \lambda < 1$ and $1 < \lambda < \lambda_{c2}$ we expect the stability of dipole-quadrupole skyrmions with predominant "magnetic" and "electric" behavior at infinity, respectively. Given $\lambda = 1$ we come to the fully $SU(3)$ isotropic solution. The critical values $\lambda_{c1,2}$ of the "anisotropy" parameter should be calculated numerically similarly to $S=1/2$ anisotropic Heisenberg model²⁰.

IV. CONCLUSION

In terms of spin coherent states we have investigated topological defects in 2D $S = 1$ (pseudo)spin quantum system with the bilinear and biquadratic isotropic exchange in the continuum limit. The proper Hamiltonian of the model can be written as bilinear on the generators of $SU(3)$ group (Gell-Mann matrices). Knowledge of such group structure enables us to obtain some new exact analytical results. The analysis of the proper classical model and its topology allows us to get different skyrmionic solutions with finite energy and the spatial distribution of spin-dipole and/or spin-quadrupole moments termed as dipole, quadrupole, and dipole-quadrupole skyrmions, respectively. Among the latter we would like to note the in-plane vortices with the in-plane distribution of spin moment, varying spin length, and the non-trivial distribution of spin-quadrupole moments.

One should note that for traditional spin systems like $3d$ magnetic oxides with $S \geq 1$ the biquadratic exchange as a rule two orders of magnitude smaller as compared with usual Heisenberg bilinear isotropic exchange. It seems, for such systems the above analysis may play only purely theoretical interest. However, for systems with the non-quenched orbital moments, (pseudo)-Jahn-Teller effect, and other forms of (pseudo)degeneracy the effective

pseudo-spin Hamiltonian given $S \geq 1$ may include different large non-Heisenberg terms. Namely for such systems we may expect the manifestation of unusual topological defects, including those addressed above.

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